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WEIGHTED L^2 APPROXIMATION OF ENTIRE FUNCTIONS

BY

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ABSTRACT. Let S be the space of entire functions $f(z)$ such that $\|f(z)\|^2 = \int\int |f(z)|^2 dm(z)$, where m is a positive measure defined on the Borel sets of the complex plane. Write $dm(z) = K(z)dA_z = K(r, \theta)dAz$. Theorem 1. *If* $\ln \inf_\theta K(r, \theta)$ *is asymptotic to* $\ln \sup_\theta K(r, \theta)$ *(together with other mild restrictions) then polynomials are dense in* S. Theorem 2. *Let* $K(z) = e^{-\phi(z)}$ *where* $\phi(z)$ *is a convex function of* z *such that all exponentials belong to* S. *Then polynomials are dense in* S.

I. **Introduction and notation.** We consider an analogue of the Bernstein problem of weighted polynomial approximation to a continuous function on the real line [3]. Let m be a positive measure defined on the Borel sets of the complex plane. We denote by S the space of entire functions $f(z)$ such that

$$\|f(z)\|^2 = \int |f(z)|^2 \, dm(z) < \infty. \tag{1}$$

Here the integration is over the complex plane. S is then a pre-Hilbert space where as usual the inner product $\langle f, g\rangle = \int f\bar{g}\, dm(z)$. We then ask when are the analytic polynomials dense in S. By "dense in S," we mean dense in the metric imposed by (1). In this paper some sufficient conditions on the measure for this to be true are given. We assume, of course, that m is such that S contains the analytic polynomials. We shall consider only the case where the measure is absolutely continuous with respect to the Lebesgue measure in the plane, i.e., $dm(z) = K(z)dA_z$.

II. We recall the known fact that when the measure is rotation invariant, i.e., $\int_E dm(z) = \int_{UE} dm(z)$ for every Borel set E and every unitary transformation U, polynomials are dense in S. Theorem 1 can be seen as an extension of this in that it generalizes the result to the case of "limited" variation.

Let

$$z = re^{i\theta}, \qquad K(z)dA_z = K(r, \theta)r\,dr\,d\theta,$$

$$K_1(r) = \inf_\theta K(r, \theta), \qquad K_2(r) = \sup_\theta K(r, \theta).$$

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THEOREM 1. *If* $K(r, \theta)$ *satisfies the following conditions, then polynomials are dense in* S.

(a) $K_1(r) = e^{-P(r)}$, $P(r)$ *positive and convex for* $r \geqslant r_1$.

(b) *For* $r \geqslant r_1$, θ *fixed,* $K(r, \theta)$ *is a decreasing function of* r *and* $\lim_{r\to\infty} K(r, \theta) = 0$.

(c) $K_2(r)$ *is bounded.*

(d) $\ln K_1(r)$ *is asymptotic to* $\ln K_2(r)$.

REMARK. We note that while conditions (a), (b), (c), and (d) are needed to ensure that polynomials belong to S and that (c) is needed for the method employed, condition (d) is our main assumption on $K(r, \theta)$.

PROOF. We may assume that $K(r, \theta) = 1$ for $0 \leqslant r \leqslant r_1$. Denote by S' the space of entire functions $f(z)$ satisfying

$$\|f(z)\|_{S'}^2 = \int |f(z)|^2 K_2(r)\, dA_z < \infty.$$

Let

$$G = \{f(\lambda z) | f(z) \in S,\ 1/2 \leqslant \lambda < 1\}.$$

One verifies easily that G is a convex subset of S. It suffices to show the following two facts.

A. G is dense in S.

B. $G \subset S'$.

Indeed $K_2(r)$ is rotation invariant; hence any function in G can then be approximated by polynomials in S'.

To prove A, note that we have pointwise convergence of $f(\lambda z)$ to $f(z)$ as $\lambda \to 1$ $(1/2 \leqslant \lambda < 1, f(z) \in S)$. Also,

$$\begin{aligned} \|f(\lambda z)\|_S^2 &= \int |f(\lambda z)|^2 K(r, \theta)\, dA_z \\ &= \frac{1}{\lambda^2} \int |f(z)|^2 K(r/\lambda, \theta)\, dA_z \leqslant 4 \int |f(z)|^2 K(r/\lambda, \theta)\, dA_z. \end{aligned} \tag{2.1}$$

But $K(r/\lambda, \theta) < K(r, \theta)$. Thus

$$\|f(\lambda z)\|_S^2 \leqslant 4M \|f(z)\|_S^2 \tag{2.2}$$

i.e., $f(\lambda z)$ is bounded in norm in S. Hence G is weakly dense in S. Since G is convex, it is in fact dense in S (see [1, p. 207]).

To prove B it is evidently sufficient to show

$$K_2(r/\lambda) < M(\lambda) K_1(r), \tag{2.3}$$

where $M(\lambda)$ is a positive constant depending only on λ. Indeed we then have

$$\|f(\lambda z)\|_{S'}^2 \leqslant 4M(\lambda)\|f(z)\|_S^2 < \infty. \tag{2.4}$$

We are assuming that $\ln K_2(r) \sim \ln K_1(r)$, i.e.,

$$K_2(r) = \exp[-P(r)Q(r)], \tag{2.5}$$

where $\lim_{r\to\infty} Q(r) = 1$, and

$$K_2(r/\lambda) = \exp[-P(r/\lambda)Q(r/\lambda)]. \tag{2.6}$$

Consider $r \geqslant r_1$ and let $A(\lambda) = r(1-\lambda)/(r-\lambda r_1)$. Since $r = A(\lambda)r_1 + (1-A(\lambda))r/\lambda$ we have, by the convexity of $P(r)$,

$$P(r) \leqslant A(\lambda)P(r_1) + (1-A(\lambda))P(r/\lambda). \tag{2.7}$$

Recall that $\lim_{r\to\infty} Q(r) = 1$. Hence given $\epsilon > 0$ $\exists R(\epsilon)$ such that $Q(r) > 1-\epsilon$ when $r > R(\epsilon)$. We choose a fixed $0 < \epsilon < A(\lambda)$. This is always possible since $A(\lambda) \geqslant 1-\lambda$ as $r \to \infty$. We then obtain

$$Q(r/\lambda) > 1 - A(\lambda) \quad \text{for} \quad r > R(\lambda). \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) we have for $r > R(\lambda)$

$$K_2(r/\lambda) < \exp[A(\lambda)P(r_1)]\exp[-P(r)] < \exp[P(r_1)]K_1(r). \tag{2.9}$$

$R(\lambda) < \infty$ and therefore $\inf_{0<r<R(\lambda)} K_1(r)$ is bounded away from 0. Hence

$$K_2(r/\lambda) < M_1(\lambda)K_1(r) \quad \text{for} \quad 0 \leqslant r \leqslant R(\lambda). \tag{2.10}$$

From (2.9) and (2.10) we get (2.3) and the desired result.

REMARK. The above theorem is readily seen to be true for several variables. We let $Z = (z_1, \cdots, z_k) = (x_1, y_1, \cdots, x_k, y_k)$ where $z_m = x_m + iy_m$, $\|Z\|^2 = r^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_k|^2$ and $dV = \Pi_{m=1}^k dx_m dy_m$. Given $dm(Z) = K(Z)dV$, we denote by S the set of entire functions in complex Euclidean k-space such that

$$\|f(Z)\|^2 = \int |f(Z)|^2 K(Z)\, dV < \infty.$$

Here the integration is over all of C_k considered as a $2k$-dimensional real euclidean space. Let $\xi_m = x_m/r$; $\eta_m = y_m/r$ and $\Theta = (\xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k)$. Then $Z = r\Theta$ and

$$\|f(Z)\|^2 = \int_0^\infty \int_{\|\Theta\|=1} |f(r\Theta)|^2 K(r\Theta)|J|\, d\Theta\, dr.$$

Theorem 1 then states the corresponding result in C_k with $\theta = \Theta$ and the proof for arbitrary k is identical to the one we have given.

III. Assume now that our measure is such that the space S contains all exponentials. It is known that the exponentials are always complete on the real line. It can be shown by example that this is not the case in the plane. In Theorem 2 we give a sufficient condition for this to be true. Let $z = x + iy$, $K(z)\,dA_z = K(x, y)\,dx\,dy$.

THEOREM 2. *Let* $K(z) = e^{-\phi(z)}$ *where* $\phi(z)$ *is a positive convex function of* z, $\phi(0) = 0$. *Assume further that the entries of the inverse Hessian matrix of* ϕ *are uniformly bounded outside a compact set containing zero and that* S *(as defined in* § I) *contains all exponentials. Then the exponentials are complete in* S.

PROOF. Since S is a Hilbert space, it suffices to show that for $b(z)$ in S

$$\langle b(z), e^{tz} \rangle = 0, \tag{3.1}$$

for all complex $t \rightarrow b(z) \equiv 0$.

The proof of (3.1) will be given as a series of lemmas. We introduce the following notation which will be used throughout the series. Let $b(z)$ be such that the left-hand side of (3.1) holds. We let

$$C(z) = b(z)e^{-\phi(z)}, \qquad \hat{d}(\alpha, \beta) = \int d(z) \exp[i\alpha x + i\beta y]\, dA_z,$$

$$\alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \text{ real},$$

$$\hat{h}(\alpha, \beta) = \hat{C}(\alpha, \beta)/(\alpha - i\beta), \qquad \psi(w) = \max_z\, (\langle z, w\rangle - \phi(z)),$$

where $\langle z, w\rangle = \operatorname{Re} z\bar{w}$ is the real inner product on R^2. Note that $\hat{C}(\alpha, \beta)$ is meaningful and an analytic function of α and β (see [4, p. 13]). Moreover by assumption

$$\int C(z) \exp[t\bar{z}]\, dA_z = \int C(z) \exp[i(-it)x + i(-t)y]\, dA_z = \hat{C}(-it, -t) = 0,$$

for all complex t. Hence $\hat{C}(\alpha, \beta) = 0$ when $\alpha = i\beta$ and therefore $\hat{h}(\alpha, \beta)$ is also analytic. Since $\phi(z)$ is convex, $\psi(w)$ is likewise convex and their relationship is a reciprocal one, namely, $\phi(z) = \max_w(\langle z, w\rangle - \psi(w))$ [2].

LEMMA 1. (a) *Let* $0 \leqslant \lambda \leqslant 1$. *Then*

$$\int |d(z)|^2 e^{\phi(\lambda z)} dA_z < \infty \longrightarrow \int |\hat{d}(\alpha, \beta)|^2 d\alpha_1 d\beta_1$$

$$< M \exp[\psi(-2\alpha_2, -2\beta_2)].$$

(b) *Let* $1/2 \leqslant \lambda < 1$. *Then*

$$\int |\hat{d}(\alpha, \beta)|^2\, d\alpha_1\, d\beta_1 < M \exp[\psi(-2\alpha_2/\lambda, -2\beta_2/\lambda)]$$

$$\longrightarrow \int |d(z)|^2 e^{\phi(\lambda' z)}\, dA_z < \infty$$

where $0 < \lambda' < \lambda$.

PROOF. By Parseval's identity we have

$$\int |\hat{d}(\alpha, \beta)|^2 \, d\alpha_1 \, d\beta_1 = \int |d(z)|^2 \exp[-2\alpha_2 x - 2\beta_2 y] \, dA_z. \tag{3.2}$$

Part a. Assume $\int |d(z)|^2 e^{\phi(\lambda z)} dA_z < \infty$, $0 \leqslant \lambda \leqslant 1$. From (3.2) we obtain

$$\int |\hat{d}(\alpha, \beta)|^2 \, d\alpha_1 \, d\beta_2 = \int |d(z)|^2 \exp[\phi(\lambda z) - 2\alpha_2 x - 2\beta_2 y - \phi(\lambda z)] \, dA_z$$

$$\begin{aligned} &< \int |d(z)|^2 \exp[\phi(\lambda z)] \exp\left[\max_z\left(\frac{-2\alpha_2}{\lambda}\lambda x - \frac{2\beta_2}{\lambda}\lambda y - \phi(\lambda z)\right)\right] dA_z \\ &= \exp\left[\psi\left(\frac{-2\alpha_2}{\lambda}, \frac{-2\beta_2}{\lambda}\right)\right] \int |d(z)|^2 e^{\phi(\lambda z)} dA_z \\ &= M \exp\left[\psi\left(\frac{-2\alpha_2}{\lambda}, \frac{-2\lambda_2}{\lambda}\right)\right]. \end{aligned} \tag{3.3}$$

Part b. Now assume

$$\int |\hat{d}(\alpha, \beta)|^2 \, d\alpha_1 \, d\beta_1 < M \exp[\psi((-2\alpha_2)/\lambda, (-2\beta_2)/\lambda)], \qquad 1/2 \leqslant \lambda < 1. \tag{3.4}$$

We multiply both sides of (3.4) by $\exp[-\psi(-2\alpha_2/\lambda', -2\beta_2/\lambda')]$, $\lambda' < \lambda$, and then integrate over the α_2, β_2 plane. Since $\psi(w/\lambda') > \psi(w/\lambda) + \text{constant}\,|w| - \text{constant}$, the right-hand side of (3.4) will be integrable and we have

$$\iiiint |d(z)|^2 \exp\left[-2\alpha_2 x - 2\beta_2 y - \psi\left(\frac{-2\alpha_2}{\lambda'}, \frac{-2\beta_2}{\lambda'}\right)\right] dx\, dy\, d\alpha_2\, d\beta_2 < M'. \tag{3.5}$$

Interchanging the order of integration we see that it suffices to show

$$\iint \exp\left[\alpha_2 x + \beta_2 y - \psi\left(\frac{\alpha_2}{\lambda'}, \frac{\beta_2}{\lambda'}\right)\right] d\alpha_2\, d\beta_2 > \text{constant } e^{\phi(\lambda' z)}. \tag{3.6}$$

By definition $\phi(\lambda' z) = \max_w (x\lambda' t + y\lambda' t - \psi(w))$ where $w = (t, t')$. Therefore in (3.6) we set $x = \psi_t(w)/\lambda'$, $y = \psi_{t'}(w)/\lambda'$ and the problem reduces to showing

$$\begin{aligned} &\iint \exp\left[\frac{\alpha_2}{\lambda'}\psi_t(w) + \frac{\beta_2}{\lambda'}\psi_{t'}(w) - \psi\left(\frac{\alpha_2}{\lambda'}, \frac{\beta_2}{\lambda'}\right)\right] d\alpha_2\, d\beta_2 \\ &\qquad > C \exp[t\psi_t(w) + t'\psi_{t'}(w) - \psi(w)]. \end{aligned} \tag{3.7}$$

Letting $\alpha_2/\lambda' = \alpha_2'$, $\beta_2/\lambda' = \beta_2'$ in the left-hand integral and then omitting the primes we see that it is enough to show

$$I = \int_{t'}^{t'+1} \int_{t}^{t+1} \exp[-(\psi(\alpha_2, \beta_2) - \psi(w) - (\alpha_2 - t)\psi_t(w) - (\beta_2 - t')\psi_{t'}(w))]\, d\alpha_2\, d\beta_2 > C. \tag{3.8}$$

When $|w| \leqslant w_1$ it is clear that the integrand in (3.8) can be bounded away from zero. Using the Taylor expansion for functions of two variables we see that it suffices to show that

$$(\alpha - t)^2\psi_{tt}(w) + 2(\alpha - t)(\beta - t)\psi_{tt'}(w) + (\beta - t')^2\psi_{t't'}(w) < M. \tag{3.9}$$

Since $t < \alpha < t + 1$, $t' < \beta < t' + 1$ and $\psi(w)$ is convex, it is certainly enough to have ψ_{tt} and $\psi_{t't'}$ bounded for $|w| > w_1$. It is here that we use the hypothesis that outside a compact set containing zero the entries in

$$\begin{pmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{yx} & \phi_{yy} \end{pmatrix}^{-1}$$

are uniformly bounded. By definition $t = \phi_x(z)$, $t' = \phi_y(z)$, and $x = \psi_t(w)$, $y = \psi_{t'}(w)$ are inverse transformations; hence the hypothesis implies that ψ_{tt} and $\psi_{t't'}$ are bounded. Upon reflection the reader will see that this is not a major restriction. This completes the proof of Lemma 1.

LEMMA 2. *Let c and c' be fixed real constants. Let $0 < \lambda < 1$. Then there exist constants M and M_1 such that*

(a) $\int |\hat{C}(\alpha_1 + ic, \beta_1 + ic')|^2\, d\alpha_1\, d\beta_1 < M \exp[\psi(-2c, -2c')]$,

(b) $\int |\hat{h}(\alpha_1 + ic, \beta_1 + ic')|^2\, d\alpha_1\, d\beta_1 < M_1 \exp[\psi(-2c/\lambda, -2c'/\lambda)]$.

PROOF. Our first assertion follows trivially from Lemma 1, since

$$\int |C(z)|^2 e^{\phi(z)} dA_z = \|b(z)\|^2 < \infty.$$

To prove (b), it suffices to show that for fixed α, β,

$$|\hat{C}(\alpha, \beta)|^2 < \text{constant} \exp[\max\{\psi(-2\alpha_2 \pm 2, -2\beta_2 \pm 2)\}], \tag{3.10}$$

where the max is taken over the four possibilities of sign. We assume (3.10) for the moment. Clearly, when either $|\alpha_1 + c'| \geqslant 1$ or $|\beta_1 - c| \geqslant 1$,

$$|\hat{h}(\alpha_1 + ic, \beta_1 + ic')| \leqslant |\hat{C}(\alpha_1 + ic, \beta_1 + ic')|.$$

From (a), we then conclude

$$\begin{aligned}(3.11)\qquad &\int_{|\beta_1-c|>1}\int_{-\infty}^{\infty} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2\,d\alpha_1\,d\beta_1\\ &+\int_{|\beta_1-c|<1}\int_{|\alpha_1+c'|>1} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2\,d\alpha_1\,d\beta_1\\ &<2e^{\psi(-2c,-2c')}.\end{aligned}$$

Now let $|\alpha_1+c'|<1$ and $|\beta_1-c|<1$. For α,β fixed, $\hat{h}(\alpha_1+iz_1,\beta_1+iz_2)$ is an entire function of z_1 and z_2. In particular it is analytic in the polydisc $|z_1-c|\leqslant 2$, $|z_2-c'|\leqslant 5$. By the maximum modulus principle we have

$$\begin{aligned}(3.12)\qquad |\hat{h}(\alpha_1+ic,\beta_1+ic')| &\leqslant |\hat{h}[\alpha_1+i(c+2e^{i\theta_1}),\beta_1+i(c'+5e^{i\theta_2})]|\\ &\leqslant |\hat{C}[\alpha_1+i(c+2e^{i\theta_1}),\beta_1+i(c'+5e^{i\theta_2})]|,\end{aligned}$$

where θ_1 and θ_2 depend, respectively, on α_1 and α_2. From (3.6) and the convexity of $\psi(w)$ we obtain

$$\begin{aligned}(3.13)\qquad &|\hat{h}(\alpha_1+c,\beta_1+ic')|^2\\ &\leqslant \text{constant}\exp\{\max\psi[-2(c+2\cos\theta_1\pm 1),-2(c'+5\cos\theta_2\pm 1)]\}\\ &\leqslant \text{constant}\exp\{\max\psi[-2(c\pm 3),-2(c'\pm 6)]\},\end{aligned}$$

where again the max is taken over the four possibilities of sign. Moreover

$$\psi(-2c\pm 6,-2c'\pm 12)\leqslant\lambda\psi\left(\frac{-2c}{\lambda},\frac{-2c'}{\lambda}\right)+(1-\lambda)\psi\left(\frac{\pm 6}{1-\lambda},\frac{\pm 12}{1-\lambda}\right)$$

and therefore

$$\begin{aligned}(3.14)\qquad &\int_{c-1}^{c+1}\int_{-c'-1}^{-c'+1} |\hat{h}(\alpha_1+ic,\beta_1+ic')|^2\,d\alpha_1\,d\beta_1\\ &\leqslant \text{constant}\exp[\psi(-2c/\lambda,-2c'/\lambda)].\end{aligned}$$

Also $\psi(w)<\psi(w/\lambda)$. Combining (3.11) and (3.14) we have (b) of Lemma 2.

To show (3.10) note that

$$\begin{aligned}(3.15)\qquad |\hat{C}(\alpha,\beta)|^2 &\leqslant \frac{1}{\pi^2}\int_{|z_1|\leqslant 1}\int_{|z_2|\leqslant 1} |\hat{C}(\alpha+z_1,\beta+z_2)|^2\,dA_{z_1}\,dA_{z_2}\\ &\leqslant \frac{1}{\pi^2}\int_{-1}^{1}dy_1\int_{-1}^{1}dy_2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\hat{C}[\alpha_1+x_1+i(\alpha_2+y_1),\\ &\qquad\qquad \beta_1+x_2+i(\beta_2+y_2)]|^2\,dx_1\,dx_2.\end{aligned}$$

Let $\alpha_1 + x_1 = \alpha_1'$, $\beta_1 + x_2 = \beta_1'$ in the last integral. From (a) and the fact that e^{ψ} is convex, we obtain (3.10).

LEMMA 3. *$C(z) = idh(z)/d(z)$.*

PROOF. By definition

$$\frac{d\hat{h}}{dz} = \int \frac{dh(z)}{dz}\, e^{i\alpha x + i\beta y}\, dA_z. \tag{3.16}$$

We integrate (3.16) by parts setting "u" $= e^{i\alpha x + i\beta y}$ and "dv" $= (dh/dz)dz$. Then "uv" $= e^{i\alpha x + i\beta y} h(z)$. We show that the boundary terms vanish. By Fubini's theorem it is sufficient to show

$$\int |e^{i\alpha x + i\beta y} h(z)|\, dA_z < \infty. \tag{3.17}$$

By Lemma 1 and Lemma 2, we have

$$\int |h(z)|^2 e^{\phi(\lambda z)}\, dA_z < \infty \qquad (0 < \lambda < 1). \tag{3.18}$$

The exponentials belong to the space. Therefore, by the Schwarz inequality

$$\int |e^{i\alpha x + i\beta y}\, h(z)|\, dA_z$$

$$\leqslant \left[\int |e^{2i\alpha x + 2i\beta y} e^{-\phi(\lambda z)}|\, dA_z\right]^{1/2} \left[\int |h(z)|^2\, e^{\phi(\lambda z)}\, dA_z\right]^{1/2} < \infty.$$

Hence,

$$d\hat{h}/dz = -\,i(\alpha - i\beta)\hat{h}(\alpha, \beta) = -\,i\hat{C}(\alpha, \beta). \tag{3.19}$$

From (3.19) we conclude that $C(z) = idh/dz$.

LEMMA 4. *$\langle b(z), g(\lambda z)\rangle = 0$ where $g(z)$ is any function in S and $1/2 \leqslant \lambda < 1$.*

PROOF. By Lemma 3, we have

$$\int C(z)\bar{g}(\lambda z)\, dA_z = i\int \frac{dh}{dz}\, \bar{g}(\lambda z)\, dA_z. \tag{3.20}$$

Integrate the right-hand side of (3.20) by parts letting "u" $= g(\lambda z)$ and "dv" $= (dh/dz)dz$. Note that $\int |\bar{g}(\lambda z)|^2 e^{-\phi(\lambda z)} dA_z < 4\|g(z)\|^2 < \infty$. Therefore, as in Lemma 3, $\int |h(z)\bar{g}(\lambda z)|\, dA_z < \infty$ and the boundary terms vanish. Since $g(z)$ is analytic $d\bar{g}/dz = 0$. It follows that

$$\langle b(z), g(\lambda z)\rangle = -\,i\int h(z)\, \frac{d\bar{g}(\lambda z)}{dz}\, dA_z = 0.$$

LEMMA 5. *Let* $f(z)$ *belong to* S, $1/2 \leqslant \lambda < 1$. *Then* $f(\lambda z)$ *converges weakly to* $f(z)$ *as* $\lambda \to 1$.

PROOF. We clearly have pointwise convergence. Moreover, $e^{-\phi(z/\lambda)} < e^{-\phi(z)}$. Hence,

$$\|f(\lambda z)\|^2 = \frac{1}{\lambda^2}\int |f(z)|^2\, e^{-\phi(z/\lambda)}\, dA_z < 4\|f(z)\|^2,$$

i.e., $f(\lambda z)$ is bounded in norm in S. The weak convergence of $f(\lambda z)$ to $f(z)$ follows (see [1, p. 207]).

Theorem 2 is now obvious. Let $g(z) = b(z)$ in Lemma 4. By Lemma 5, $b(\lambda z)$ converges weakly to $b(z)$ and $\|b(z)\| = 0$.

COROLLARY. *Polynomials are dense in the space* S.

REMARK. Theorem 2 was proven by different methods by B. A. Taylor in [5] with slightly different conditions on $\phi(z)$. The theorem is given there for the general n-variable case. Our proof is simpler and more direct for the single-variable situation, but unfortunately it does not generalize.

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